Dispersion requirements of low-loss negative refractive index materials and their realisability

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Abstract

A necessary dispersion requirement is given for the effective relative permeability, $\mu_r(\omega)$ and permittivity $\epsilon_r(\omega)$ as a function of frequency for lossy homogeneous dispersive materials and negative refractive index materials in particular, if they exist at some dimensional scale D. The requirements are valid both for isotropic and uniaxial materials, the latter defined by material properties that are invariant under a rotation about a principal axis. Using asymptotic requirements at zero frequency and Foster's reactance theorem, the general band-gap structure is given for low-loss materials. It is shown that such materials can only be achieved if the material exhibits finite band-gaps separating regions of positive and/or negative refractive index regions. In degenerate cases the band gaps shrink to zero but low-loss propagation remains impossible in the neighbourhood of the critical frequencies. The effect of small losses are considered, and compared with Stockman's criterion. Losses may be arbitrarily small within a band provided there are no bounds to the size of the real parts of the relative permittivity and permeability at the band edges. A numerical example is given showing the effect of small losses.

1 Introduction

To the author's knowledge no negative dielectric (three dimensional) materials have yet been demonstrated that exhibit negligible loss and which are homogeneous on a small scale. There are good reasons for this. Recently Stockman [1] has shown that negative refractive index materials require the existence of losses and impose bounds on the minimum necessary imaginary parts of the effective relative permittivity ϵ_r and permeability μ_r over the band of interest. If Stockman's integral at an angular frequency ω is defined by,

$$I_s = \frac{2}{\pi}\int_0^\infty \frac{-\epsilon_r''(\omega_1)\mu_r'(\omega_1) - \mu_r''(\omega_1)\epsilon_r'(\omega_1)}{(\omega_1^2-\omega^2)^2}\omega_1^3\,d\omega_1 \tag{1}$$

then Stockman's necessary requirement is that $I_s \leq -1$ for negative refractive index materials, where $\epsilon_r' = \Re(\epsilon_r)$, $\epsilon_r'' = \Im(\epsilon_r) \leq 0$, $\mu_r' = \Re(\mu_r)$ and $\mu_r'' = \Im(\mu_r) \leq 0$, where we assume

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an $e^{j\omega t}$ harmonic time convention (this defines losses with negative imaginary parts to the relative permeability and permittivity and hence the sign change in (1) since Stockman uses the alternative $e^{-i\omega t}$ convention). This condition may also be true for a material for which either ($\mu_r' > 0$ AND $\epsilon_r' < 0$) or ($\mu_r' < 0$ AND $\epsilon_r' > 0$).

This inequality necessarily implies that $\mu_r'' < 0$ and/or $\epsilon_r'' < 0$ over at least part of the spectrum. Stockman further concludes that the losses must be significant in the neighbourhood of a given frequency in the negative refractive index frequency band.

It is worth pointing out, however, that the latter conclusion is valid only if the real parts, ϵ_r' and μ_r' are bounded everywhere. This will obviously be true of real materials, where there are always small but non-zero losses, but there is no *a-priori* reason why materials should not exist with finite but very large values for ϵ_r' and/or μ_r'.

We also show that I_s, even when evaluated in principal value sense, is not always bounded for a realisable positive material. If I_s can also be unbounded for a negative material, Stockman's inequality may equally well be replaced by the inequality $I_s < 0$ with no loss of information. In this case, the inequality conveys no non-trivial information concerning the size of the required losses.

This paper seeks to tackle the problem of dispersive media, and negative refractive index materials, in a different manner. We apply equivalent circuit theory to model realisable dispersion since it is always possible to construct an artificial material using a suitably large number of lumped impedance elements; in general comprising resistors, inductors and capacitors. Such an approach is physically valid provided the scale length of the lumped elements is sufficiently small compared to a wavelength that the material behaves homogeneously on a scale small but not very small compared to a wavelength [11]. We will term such a material 'pseudo-homogeneous' to distinguish from an idealised homogeneous material where the actual granularity is usually considered on the molecular scale.

We consider both isotropic and uniaxial materials where a scalar transmission line model is valid. Isotropic materials may either be amorphous or represented by a periodic crystal structure with a cubic unit cell. A uniaxial material can be constructed from a layered material comprising stacked frequency selective surfaces (FSS) under certain conditions; for example, (1) each FSS of the stack is laterally isotropic, e.g. possessing 3- 4- or 6-fold rotation symmetry with negligible evanescent mode coupling between contiguous FSS as $D \to 0$. In this case, each FSS does not need to share the same unit cell. (2) there exists a global unit cell where the global unit cell is 3-, 4- or 6-fold rotationally symmetric. The second condition is well known [2] and is the most common requirement for a uniaxial crystal structure. To be exact, the first condition requires that each FSS has a unit cell that is small compared to the distance between FSS, both bounded by D as $D \to 0$. This assures negligible evanescent mode interaction, in which case lateral displacements and rotations of one FSS with respect to another have no effect on the electromagnetic interactions between layers.

For materials of this type there is a formal identity between the transmission line equations and the wave equations in a homogeneous medium provided there exists a scale length D, defining the granularity of the lumped impedance elements, which is small compared to a free space wavelength. Note that in a uniaxial material the effective permittivity and permeability will be a function of the angle of incidence and the polarisation (TE or TM) of the wave [2]. The transmission line model may also be obtained as the low frequency limit of a general Floquet modal vector field expansion (e.g. [3]). This would suggest that there is a way of connecting the one-dimensional transmission line model to a physical representation of a more general biaxial material, but we do not consider this here.

Provided quantum mechanical effects are not important in describing the electron interactions with fields, a lumped impedance element model can provide a complete formulation of Maxwell's equations using effective permittivities and permeabilities as defined (for the scalar linear case) below. Under these assumptions, a loss-less lumped element material requires the absence of resistors. Conversely, an idealised (but in principal realisable) almost loss-less dispersive material can be constructed using a three dimensional network of just capacitors and inductors.

We can say nothing of how the macroscopic Maxwell equations must be modified for the inclusion of general quantum mechanical interactions. In general, non-local and non-linear effects may be required resulting in constitutive relations with general spatial/temporal dispersion as well as non-harmonic representations of the fields. The theory of superconductors is a case in point [12].

The principal aim of this paper is to show the general dispersive properties of a realisable pseudo-homogeneous artificial material and, in particular, how this relates to negative refractive index materials. The findings are quite consistent with Stockman's results and causality. The motive is to set out, in circuit equivalent terms, how negative refractive index materials sit within the context of general dispersion, whether they are realisable, and the nature of the losses that must be present.

The lumped element representation also shows that the scale length D (more accurately, D/λ, where λ is the free space wavelength) cannot be reduced to zero, for a negative refractive index material, without the existence of infinite value capacitors and inductors occupying a vanishingly small space. When materials are not gyromagnetic, these elements are not just mathematical abstractions and represent the capacitors and inductors that must be inserted or synthesised within an artificial material to achieve the necessary dispersion. For example as inductors between conducting patches on a stacked array of frequency selective surfaces or as capacitors between such arrays, as illustrated later. The requirement for indefinitely large capacitors and inductors places physical limits on the degree of homogeneity that can be achieved before a negative refractive index is impossible in such materials.

When materials include gyromagnetic interactions, dispersion in the magnetic permeability can be modelled with infinite value capacitors because the underlying dynamic model does not require charge storage as a result of Maxwell's equations. Instead, the magnetic dynamics are usually modelled using the Landau-Lifshitz-Gilbert equation (or generalisations thereof) which takes a different form and a circuit model using capacitors becomes just a mathematical abstraction. However, this is not true of dispersion models for the electric permittivity and we are aware of no physical mechanism for a realisation of infinite value inductors in the homogeneous limit. Since negative refractive index materials require both infinite capacitors and infinite inductors in the homogeneous limit, it is difficult to see how negative refractive index materials can exist here.

2 Stockman's integral

Stockman's integral, I_s, defined in equation (1) above, must be evaluated in principal value sense avoiding the singularity when $\omega = \omega_s$. To gain some further insight, let us define

$$f(x) = -\epsilon_r''(x)\mu_r'(x) - \mu_r''(x)\epsilon_r'(x) \tag{2}$$

The Cauchy principal value integral is defined by

$$I_s = \lim_{\epsilon \to 0} \left(\int_0^{\omega-\epsilon} + \int_{\omega+\epsilon}^{\infty} \right) \frac{f(x)x^3\,dx}{(x^2-w^2)^2} \tag{3}$$

Under a change of variables $x = \omega \tan\theta$,

$$I_s = \lim_{\delta \to 0} (I_L + I_U) \tag{4}$$

where

$$I_L = \frac{2}{\pi} \int_0^{\pi/4-\delta} \frac{f(\omega\tan\theta)\sin^3\theta\,d\theta}{\cos\theta\cos^2 2\theta} \tag{5}$$

$$I_U = \frac{2}{\pi} \int_{\pi/4+\delta}^{\pi/2} \frac{f(\omega\tan\theta)\sin^3\theta\,d\theta}{\cos\theta\cos^2 2\theta} \tag{6}$$

for positive non-zero $\delta = \epsilon/(2\omega)$. We now make two assumptions; firstly that $f(x)$ and all its derivatives are everywhere bounded (which requires losses) and secondly that the effect of the losses is sufficiently small as $x \to \infty$ so that $f(\omega\tan\theta) \to O(\cos^t\theta)$ for $t > 0$ as $\theta \to \pi/2$. Both these assumptions are taken as valid if the poles of the functions $\epsilon_r(s)$ and $\mu_r(s)$ (see below) are confined to a bounded region in the left half complex s-plane. In this case, I_L and I_U are bounded integrals for $\delta \neq 0$.

However, there is no guarantee that the principal value exists. For example, consider a positive lossy realisable material defined by $\mu_r(\omega) = 1$ and $\epsilon_r(\omega) = 1 - j\sigma/(\varepsilon_0\omega)$ where σ is the zero frequency conductivity of the material and ε_0 is the permittivity of free space. In this case we find that I_s is given by the principal value integral,

$$I_s = \frac{\sigma}{\pi\varepsilon_0\omega} \mathcal{P} \int_0^{\pi/2} \frac{d\theta}{\cos^2 2\theta} \tag{7}$$

The integrand is positive everywhere and symmetric about $\theta = \pi/4$ so that $I_L = I_U$ and hence I_s is positive infinite.

Clearly I_s must take a finite negative value if Stockman's inequality $I_s < -1$ is to hold any more meaning than $I_s < 0$, which would convey no useful information about the size of the required losses. Unfortunately, to address this we need an example of a material with a realisable negative refractive index with an analytic functional form for $f(\omega)$. A numerical example is provided later, where we employ our general dispersive forms to construct such an $f(\omega)$ and attempt to evaluate I_s numerically. However, although our example satisfied Stockman's inequality we were not able to demonstrate convergence to a finite value. No special numerical methods were employed, though, so the result is not conclusive.

3 Principal assumptions and fundamentals

In this article use is made of the following (possibly related) assumptions:

1. The material is three dimensional linear, passive, isotropic or uniaxial and homogeneous at a suitable dimensional scale, D.
2. All materials obey the fundamental laws of thermodynamics.
3. All electromagnetic fields in the materials satisfy Maxwell's equations.
4. The relative permeability at zero frequency is finite.
5. There are no quantum mechanical interactions required to form the constitutive relations in Maxwell's equations.

Assumption 1, and its implications, is fairly clear and has been discussed in the introduction. The latter assumptions are discussed as context demands.

Under assumptions 1, 3 and 5 and at a certain length scale D, any inhomogeneities present at a sufficiently finer scale can be modelled by an effective relative permittivity ϵ_r and relative permeability μ_r which are scalar quantities. The Maxwell equations can assume our harmonic form and may be given by,

$$\begin{array}{ll} \nabla.\underline{H} = 0 & \nabla.\underline{E} = \rho \\ \nabla \times \underline{H} = \underline{J} + j\omega\varepsilon_0\overline{\epsilon}_r\underline{E} & \nabla \times \underline{E} = -j\omega\mu_0\mu_r\underline{H} \\ \underline{J} = \sigma\underline{E} & \end{array} \tag{8}$$

Also under our assumptions, a linear isotropic relationship exists between the electric current $\underline{J}$ and the electric field $\underline{E}$. The nomenclature is SI units standard with $\mu_0 = 4\pi \times 10^{-7}$ Henry/m and $\varepsilon_0 = 8.854 \times 10^{-12}$ farad/m. The "true" relative permittivity, $\overline{\epsilon}_r$ and the conductivity σ are combined as an effective relative permittivity,

$$\epsilon_r(\omega) = \frac{\sigma(\omega)}{j\omega\varepsilon_0} + \overline{\epsilon}_r(\omega) \tag{9}$$

Under this convention *define* $\sigma(\omega) = \sigma_0$ as a real constant independent of frequency and subsume any additional frequency dependence in the complex function $\overline{\epsilon}_r$. Since the original time-dependent quantities in Maxwell's equations are real,

$$\epsilon_r(\omega) = \epsilon_r^\star(-\omega) \qquad \mu_r(\omega) = \mu_r^\star(-\omega) \tag{10}$$

for real ω, where the * represents complex conjugation.

4 General forms for ϵ_r and μ_r near zero frequency

Any dispersive pseudo-homogeneous material, if one exists at a dimensional scale D, must have a characteristic impedance function $K(s)$, writing $s = j\omega$, which is realisable as a ladder network whose ladder elements may be represented in terms of a network of resistors, capacitors and inductors. In an isotropic material, the ladder network represents the isotropic propagation of a plane wave, as a function of frequency and distance, in any direction in the material. In a uniaxial material, the network component values and thus the effective permittivity and permeability assume a certain direction of propagation of the wave. A physical realisation of such a material (e.g. as a stacked array of frequency selective surfaces) shows a variation of these properties with angle of incidence and polarisation.

Solution of the wave equation using such a representation is performed using the per unit length series impedances, $Z(s)$ in $\Omega\, m^{-1}$ and per unit length shunt reactances $Y(s)$ in $\Omega^{-1}\, m^{-1}$ (see, for example [7], section 7.4) where $Z(s)$ and $Y(s)$ are real functions of s. The ladder network representation is shown in figure 1, below, where the lumped element representation is explicit. Over a dimensional scale distance D, the lumped impedances and admittances have values $DZ(s)$ and $DY(s)$. In the limit of a perfectly homogeneous material it is required that $D \to 0$. The characteristic impedance $K(s)$ and wave number $k(s)$ have the following

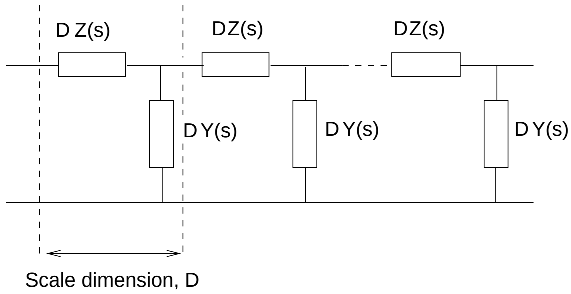

Figure 1: *A uniform transmission-line representation of a homogeneous material*

equivalences;

$$K(s) = \sqrt{\frac{Z(s)}{Y(s)}} = Z_0 \sqrt{\frac{\mu_r(s)}{\epsilon_r(s)}} \tag{11}$$

$$k(s) = \pm j\sqrt{Z(s)Y(s)} = \mp \frac{js}{c_0}\sqrt{\mu_r(s)\epsilon_r(s)} \tag{12}$$

where c_0 is the speed of light in free space and Z_0 is the impedance of free space. Solution of these equations gives,

$$\mu_r(s) = \pm \frac{c_0}{Z_0\, s} Z(s) \tag{13}$$

$$\epsilon_r(s) = \pm \frac{c_0\, Z_0}{s} Y(s) \tag{14}$$

Since the real parts of $Z(s)$ and $Y(s)$ are positive for real frequencies and the imaginary parts of μ_r and ϵ_r are negative, we must take the plus sign in both expressions.

Let us first consider the properties of the relative permittivity. In general equation (10) implies that the real part, $\Re(\epsilon_r)$ is an even function of ω and the imaginary part, $\Im(\epsilon_r)$ is an odd function of ω. In addition,

$$\Re(\epsilon_r) = O(\omega^{-2}) \ \ \text{, and } \Im(\epsilon_r) = O(\omega^{-1}) \quad \text{as } \omega \to 0 \tag{15}$$

This is necessary in order that the admittance $Y(s)$ has no greater than a first order pole at the origin, required for any realisable impedance or admittance function [9]. Furthermore assumption 2 implies [6] that when the effect of losses is negligible, such as where the $O(\omega^{-2})$ term dominates as $\omega \to 0$, $d(\omega\epsilon_r(\omega))/d\omega > 0$ and $d(\omega\mu_r(\omega))/d\omega > 0$. The former implies that the coefficient of the $O(\omega^{-2})$ term must be negative (or zero). Further, for a passive material $\Im(\epsilon_r) \leq 0$ for all real ω. This implies the following general expansion for $\epsilon_r(\omega)$ as $\omega \to 0$,

$$\epsilon_r(\omega) \sim [-\frac{\epsilon_{-2}^{(r)}}{\omega^2} + \epsilon_0^{(r)} + \epsilon_2^{(r)}\omega^2 + \epsilon_4^{(r)}\omega^4 + \ \ldots] + j[-(\sigma_0/\varepsilon_0).\frac{1}{\omega} + \epsilon_1^{(i)}\omega + \epsilon_3^{(i)}\omega^3 + \ldots] \tag{16}$$

with all coefficients defined real. The first term represents the “free electron plasma” (e.g. section 10 of [12]) term with a coefficient $\epsilon_{-2}^{(r)} \geq 0$. We also have $\sigma_0 > 0$. Also, if $\epsilon_{-2}^{(r)} = 0$ then it may be shown under assumption 2 (see section 14 of [6]) that $\epsilon_0^{(r)} \geq 1$.

The inclusion of the plasma term is important if we wish to include the modelling of a loss-less admittance function $Y(s)$ which is inductive near zero frequency. If series losses are present (as is the case for real inductors), there is no pole at the origin and this term is strictly speaking absent.

Similar arguments may be applied to the relative permeability, with some important differences. Firstly, the non-existence of magnetic monopoles implies there can be no magnetic current at zero frequency, so there is no $O(1/\omega)$ term. The absence of magnetic monopoles also implies there is no classical mechanism for a zero frequency ‘magnetic plasma’ term; i.e. there is no mechanism which involves the generation of magnetic fields by Faraday induced electric currents. This does not rule out some as-yet undiscovered quantum-mechanical effect, but this would be exceptionally remarkable. We will therefore assume (assumption 4), even in the absence of losses, that the relative permeability at zero frequency is always finite. Any physically realisable magnetic plasma must have a non-zero low-frequency band edge (e.g. as given in [5]), consistent with our formulation.

$$\Re(\mu_r) = O(1) \ \ \text{, and } \Im(\mu_r) = O(\omega) \quad \text{as } \omega \to 0 \tag{17}$$

and μ_r may be expanded near zero frequency as,

$$\mu_r(\omega) \sim [\mu_0^{(r)} + \mu_2^{(r)}\omega^2 + \mu_4^{(r)}\omega^4 + \ \ldots] + j[\mu_1^{(i)}\omega + \mu_3^{(i)}\omega^3 + \ldots] \tag{18}$$

where $\mu_1^{(i)} \leq 0$ for a passive material, $\mu_0^{(r)} > 0$ (less restrictive than the requirement on $\epsilon_0^{(r)}$, see sections 31 and 32 of [6]) and all coefficients real.

Any realisable lossy driving point impedance or admittance function can be represented (e.g. [9]) as a rational function with poles and zeros in the left half of the complex s-plane. Such a representation may be applied, individually, to the elements $Z(s)$ and $Y(s)$ so that,

$$Z(s) = \frac{\beta_1(s)}{D} \frac{\prod_{i>0}(1 + js/p_i^{(1)})(1 - js/p_i^{(1)\star})}{\prod_{i>0}(1 + js/q_i^{(1)})(1 - js/q_i^{(1)\star})} \tag{19}$$

$$Y(s) = \frac{\beta_2(s)}{D} \frac{\prod_{i>0}(1 + js/q_i^{(2)})(1 - js/q_i^{(2)\star})}{\prod_{i>0}(1 + js/p_i^{(2)})(1 - js//p_i^{(2)\star})} \tag{20}$$

where the '$\star$' represents the complex conjugate, $jp_i^{(1)}$ and $jq_i^{(1)}$ are respectively the complex zeros and poles of $Z(s)$ and $jp_i^{(2)}$ and $jq_i^{(2)}$ are respectively the complex zeros and poles of $1/Y(s)$. The $\beta_1(s)$ and $\beta_2(s)$ terms take one of three possible forms,

$$\beta_{1,2}(s) = \begin{cases} b_{1,2}^{(0)} & \text{if there is loss at zero frequency} \\ b_{1,2}^{(1)}s & \text{if } Z(s) \text{ is inductive, } Y(s) \text{ is capacitive} \\ b_{1,2}^{(2)}/s & \text{if } Z(s) \text{ is capacitive, } Y(s) \text{ is inductive} \end{cases} \tag{21}$$

with all coefficients real and positive. In general, for general passive impedances/admittances, the forms for $Z(s)$ and $Y(s)$ are realisable as long as $\Re(Z(s)) \geq 0$ and $\Re(Y(s)) \geq 0$ for all real frequencies ω.

For representing realisable materials, (16), (18), (13) and (14) imply that $\beta_1(s)$ can take only the form $\beta_1(s) = b_1^{(1)}s$, whereas $\beta_2(s)$ may take any of these three forms. Consequently in general we have the permitted forms,

$$\mu_r(\omega) = \frac{c_0}{Dp_0^{(1)}} \frac{\prod_{i>0}(1 - \omega/p_i^{(1)})(1 + \omega/p_i^{(1)\star})}{\prod_{i>0}(1 - \omega/q_i^{(1)})(1 + \omega/q_i^{(1)\star})} \tag{22}$$

$$\epsilon_r(\omega) = \frac{c_0 g(\omega)}{D} \frac{\prod_{i>0}(1 - \omega/q_i^{(2)})(1 + \omega/q_i^{(2)\star})}{\prod_{i>0}(1 - \omega/p_i^{(2)})(1 + \omega/p_i^{(2)\star})} \tag{23}$$

where $g(\omega)$ takes one of three possible forms;

$$g(\omega) = \begin{cases} 1/q_0^{(2)} & \text{No loss at zero frequency, no plasma term.} \\ -jZ_0\sigma_0^{(2)}/\omega & \text{Loss present at zero frequency.} \\ q_{-2}^{(2)}/\omega^2 & \text{No loss at zero frequency with plasma term.} \end{cases} \tag{24}$$

where the coefficients $p_0^{(1)}$, $q_0^{(2)}$, $\sigma_0^{(2)}$ and $q_{-2}^{(2)}$ are real and positive.

Note that the analytic forms for $\epsilon_r(\omega)$ and $\mu_r(\omega)$ are quite general and represent realisable functions with the assumed positivity condition. They are, however, only *necessary* conditions and not *sufficient* in the sense that we have enforced no conditions in the high frequency limit. The latter because the impedance/admittance model is only valid up to some finite upper frequency limit requiring $\omega \ll 2\pi c_0/D$. The expressions may also be interpreted as representations of the electric and magnetic susceptibility defined by a set of damped harmonic dipole oscillators; electric dipoles for $\epsilon_r(\omega)$ and magnetic dipoles for $\mu_r(\omega)$. This latter approach has been applied for single resonators and suggested for multiple resonators (through a summation of multiple Lorentz dispersion characteristics), in order to obtain physically realisable dielectric materials [14] and characteristics such as the energy density [13].

5 Dispersion when losses are small

We may define $p_i^{(1,2)} = p_i^{(1,2)\prime} + jp_i^{(1,2)\prime\prime}$ and $q_i = q_i^{(1,2)\prime} + jq_i^{(1,2)\prime\prime}$ in terms of their real and imaginary parts. When losses are sufficiently small, p_i and q_i have negligible imaginary part and we may apply Foster's reactance theorem. One form of this theorem has been applied [8] to show that the input reactance of a loss-less negative refractive index material terminated by a perfect conductor has a frequency derivative greater than zero. We apply this theorem in a different manner; not to a terminated input impedance but to the individual series and parallel terms of the ladder network representation. Thus in order for the rational functions to be positive real, Foster's theorem states that the poles and zeros of $Z(s)$ and $Y(s)$ for a loss-less material must occur on the real frequency axis (imaginary s- axis) and interlace each other. This is necessary and sufficient for the impedance/admittance functions to be passive and realisable (see, e.g. section 9.4 of [9]). Evidently this theorem can be extended when sufficiently small losses are present in which case for a passive realisable impedance the poles and zeros occur either on the real frequency axis or near it in the left half of the complex s-plane. This is because the forms (22) and (23) imply that the position of the poles and zeros do not move discontinuously as a function of a loss parameter (such as the loss tangent at a non-polar frequency), as this parameter changes continuously from zero. A similar argument is made in Bode [9], (section 10.11). Provided the losses are small enough, the interlacing of the imaginary parts of the s-plane poles and zeros remains as the loss-less case.

It is observed that as $D \to 0$, the factors to the left of the ratios of products must remain finite for finite relative permeability and permittivity for all except a possible discrete set of frequencies defined by any poles on the real frequency axis. This relates to an important matter which will be discussed later.

When all loss terms are small, the interlacing property of the remaining coefficients p_i and q_i is, for the permeability function,

$$0 < q_1^{(1)\prime} < p_1^{(1)\prime} < q_2^{(1)\prime} < p_2^{(1)\prime} < q_3^{(1)\prime} < p_3^{(1)\prime} \dots \tag{25}$$

and separately one of two possible cases for the permittivity function,

$$\begin{array}{ll} 0 < p_1^{(2)\prime} < q_1^{(2)\prime} < p_2^{(2)\prime} < q_2^{(2)\prime} < p_3^{(2)\prime} < q_3^{(2)\prime} \dots & \text{when } q_{-2}^{(2)} = 0 \text{ (no plasma term)} \\ 0 < q_1^{(2)\prime} < p_1^{(2)\prime} < q_2^{(2)\prime} < p_2^{(2)\prime} < q_3^{(2)\prime} < p_3^{(2)\prime} \dots & \text{when } q_{-2}^{(2)} \neq 0 \text{ (plasma term present)} \end{array} \tag{26}$$

For a zero (or low) loss material $\mu_r(\omega)/\epsilon_r(\omega)$ is (almost) real over all (most) frequencies and may be either positive or negative. If negative, no plane wave can propagate in the medium since $K(\omega)$ is imaginary. For a positive material, both $\mu_r'(\omega) > 0$ and $\epsilon_r'(\omega) > 0$. Conversely, for a negative refractive index material, $\mu_r'(\omega) < 0$ and $\epsilon_r'(\omega) < 0$.

The poles and zeros of (22) and (23), when taken together, permit either one or two consecutive poles along the real frequency axis (or near this axis, for a material with small but non-zero loss). Conversely, there must be either one or two consecutive zeros before the next pole. For small but non-zero losses, the first critical point for both $\mu_r(\omega)$ and $\epsilon_r(\omega)$, as ω is increased from zero, must be a pole. If there is zero loss at zero frequency, $\epsilon_r(\omega)$ may have a pole at the origin so in this special case the first critical point as ω is increased from zero is a zero.

Figure 2 shows the generic behaviour for a perfectly loss-less material with no plasma term, where the essential presence of forbidden bands is apparent, separating the regions of positive and negative refractive index. Similar behaviour is apparent if there is a plasma term, but where the curve for ϵ_r is shifted to the left by one critical point. If there are two consecutive poles or two consecutive zeros the material switches from a positive to a negative material (or *vice-versa*) after the last band gap. It is apparent that the necessary dispersion rules, $d(\omega\epsilon_r(\omega))/d\omega > 0$ and $d(\omega\mu_r(\omega))/d\omega > 0$ (e.g. section 80 of [6]) for passive loss-less materials are satisfied everywhere. Figure 3 shows the resultant behaviour of the impedance function $K(\omega)$ and wave number $k(\omega)$ in the loss-less case.

When the materials are slightly lossy, the poles and zeros are moved away from the real frequency axis into the left hand half of the complex s-plane. This results in a small shift of the position of the zeros of the real part of $\epsilon_r(\omega)$ and $\mu_r(\omega)$. The infinities, on the other hand, vanish with a zero crossing of the real part. Near the vanishing of the poles, the small losses are always large enough so that $d(\omega\Re[\epsilon_r(\omega)])/d\omega \not> 0$ and $d(\omega\Re[\mu_r(\omega)])/d\omega \not> 0$. In this special sense, it is clear that the presence of a non-vanishing loss (non-zero imaginary part to p_i or q_i) always results in a significant difference in the values of ϵ_r and/or μ_r in the neighbourhood of these critical frequencies. Conversely, any bounds set on the maximum magnitudes of the real parts of ϵ_r and/or μ_r must be inversely related to the size of the neighbourhoods for which losses are significant. Some analytic properties of the functions, when there is small but non-zero loss, are given later.

Special behaviour is obtained if there are common poles or zeros, $p_i^{(1)} = q_i^{(2)}$ or $q_i^{(1)} = p_i^{(2)}$ for some i. This results in a disappearance of a finite width forbidden band. However, when there are small losses the impedance still tends to zero or infinity at the critical points resulting in a narrow (open) stop band in the neighbourhood of the pole or zero. The extreme case is where $p_i^{(1)} = q_i^{(2)}$ and $q_i^{(1)} = p_i^{(2)}$ for all i, in which case $Z(s) = \alpha Z_0^2 Y(s)$ for some positive constant α; i.e. the elements $Z(s)$ and $Y(s)$ are network complements with $\epsilon_r(\omega) \propto \mu_r(\omega)$. This is not possible if there is a dielectric plasma term or if there is loss at zero frequency. Nor is it possible over an indefinitely large frequency band (see figure 4, below). In fact it is apparent, using (22) and (23), or the low frequency asymptotic forms for $\epsilon_r(\omega)$ and $\mu_r(\omega)$ on which they are based, that certain common models, such as the *dual transmission line model* (where $Z(\omega)$ is purely capacitive and $Y(\omega)$ is purely capacitive) are not physically possible for a three dimensional material. This is rather unfortunate since it has been shown that such a model, valid for one dimensional transmission lines, has optimum bandwidth [4].

In terms of the equivalent circuit ladder model, the low frequency asymptotic form requires that $Z(s)$ should behave like an inductor and $Y(s)$ like a capacitor with a possible shunt resistance or, if there is a plasma term, like a shunt inductor. Although, given the model limitation of non-zero D, we have not imposed any high frequency behaviour it is known in this limit that $\epsilon_r \to 1$ and $\mu_r \to 1$ (sections 78-79 of [6]). This implies that $Y(s)$ must

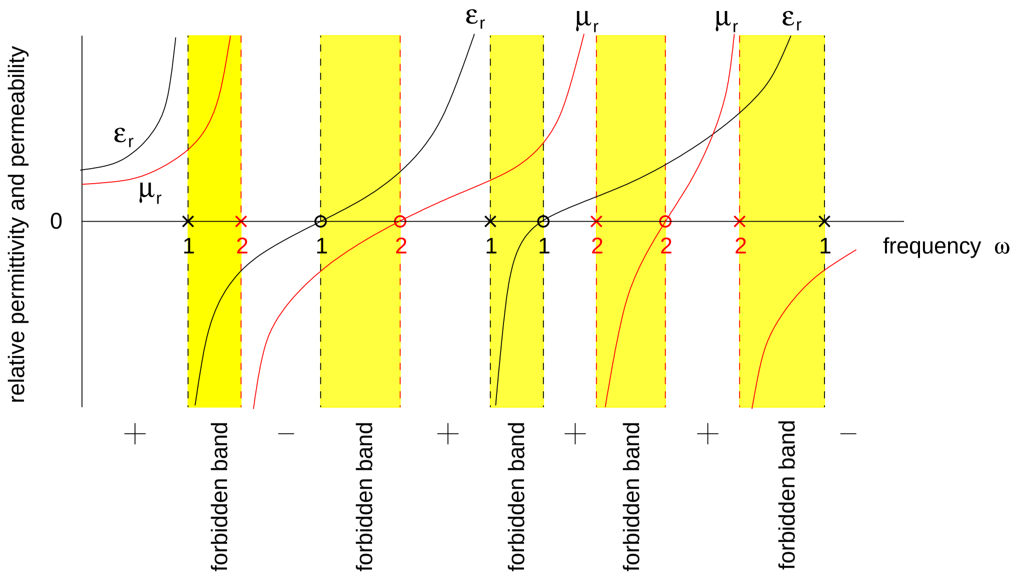

Figure 2: *Generic dispersive behaviour for a loss-less material*

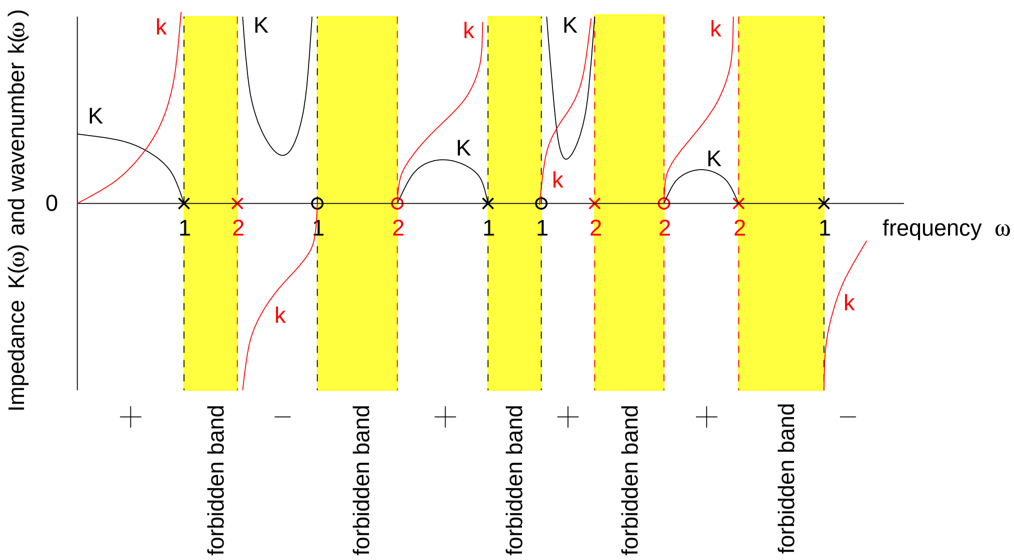

Figure 3: *Generic behaviour of the impedance and wave function for a loss-less material*

look capacitive at high frequencies and thus, without any loss of generality and to maintain a symmetry of form, a suitable ladder network representation of a general uniaxial or isotropic material is shown in figure 4, where the parallel impedance elements Z_1' contains no parallel inductive element and Z_2' contains no parallel capacitive element. The capacitor C_2 represents the capacitance in the medium at the highest frequency for which the model is valid, whereas the inductor L_1 represents the inductance at zero frequency,

$$C_2 = \frac{D\Re(\epsilon_r(\omega_{max}))}{Z_0 c_0} \quad \text{and} \quad L_1 = \frac{Z_0 D \mu_r(0)}{c_0} \tag{27}$$

Here it is only required that $\Re(\epsilon_r(\omega_{max})) \geq 0$, not that this should be greater than unity. Apart from the non-degeneracy requirement on Z_1' and Z_2', these elements are free to represent any circuit combination of resistors, capacitors and inductors (which guarantees the positivity requirement) in the representation of an arbitrary material from zero to any finite upper frequency bound. Other equivalent circuit forms are possible, but ours is the simplest that permits an (in principle) arbitrary realisable network impedance (Z_1' or Z_2') over a finite frequency range. This form may not, however, minimise the component count necessary to model a given material. For example in [10], a minimal component count equivalent circuit is given where our $Z_1' || sL_1$ form is replaced by an inductor in series with an inductor-||-resistor-||-capacitor in parallel with combined impedance $Z_s + (1/sC_r \,||\, sL_r \,||\, R_r$ for a split ring resonator/wire medium. However, in this formulation it is not possible to have a medium for which $L_r \to \infty$ unless (trivially) $R_r = 0$.

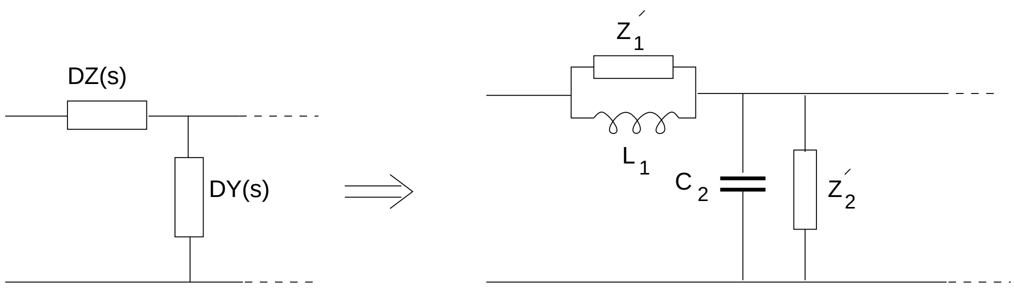

Figure 4: *Representation of a homogeneous material valid in the low frequency limit*

6 Physical representations

It is possible to identify the lumped impedance values in the circuit model with the physical components necessary to synthesise a real structure in both laterally isotropic and isotropic materials if there are no gyromagnetic materials present. In this case the physical mechanism for dispersion arrise purely from Maxwell's equations and local non-dispersive constitutive relations at a microscopic scale. If gyromagnetic materials are present the lumped series impedance term $Z(s)$ does not necessarily require physical capacitors and inductors that store electric current and charge. This is because the physical mechanisms responsible for dispersion of the permeability in such materials are not determined by Maxwell's equations but by gyromagnetic dynamics; usually modelled using the Landau-Lifshitz-Gilbert equations or variations thereof. However, dispersion of the permittivity is still governed by Maxwell's equations and local non-dispersive constitutive relations at a microscopic scale. This is true even if there are gyromagnetic materials present and so the lumped parallel admittance $Y(s)$ must still be realised in terms of the operation of physically real capacitors and inductors requiring local charge and current storage mechanisms.

Uniaxial materials can be realised as a stacked array of frequency selective surfaces with (in general) interconnections between layers. The representation is well known in one form or another, but will be shown here to clarify the nature of the components. The representation of isotropic materials needs a little more explanation.

Let us first consider the uniaxial material, with relative permittivity and permeability that are a function of angle of incidence and polarisation. The L_1 and C_2 parts of the component model in figure 4 describe a natural 'embedding' representing the permeability of the underlying material at low frequency and the permittivity at high frequency. There are various methods to realise the remaining components, one of which is illustrated in figure 5 (showing a section of a multiple stacked array). It is assumed that the greater of the distance between arrays and the unit cell size is given by D. Essentially, the shunt admittance Z_2' components may be realised by the use of an array of conducting patches with discrete capacitors and inductors (or their on-surface mini-scale fabrications) joining contiguous patches. The series impedance Z_1' components may be realised by the use of discrete or synthesised capacitors and inductors between patches of one array and the next. With gyromagnetic materials present these latter components need not be synthesised using real capacitors and inductors. Although shown between layers, the inter-planar components can be mounted co-planar with simpler wire vias between surfaces for ease of construction. Actually, vias can be "synthesised" out of further sets of thin FSS, where specific use is made of evanescent mode interactions. For example, as illustrated, a 'column' of closely spaced capacitively coupled patches designed so that the capacitance between patches in a column is as large as necessary.

To provide a physical representation of an isotropic material, consider a cubic unit cell of side length D. We need to consider *two* interlaced cubic lattices where the nodes of one lattice lie at the mid-points of the nodes of the other, similar to the structure required by the Yee FDTD algorithm [15]. This is shown in figure 6, where the nodes of the primary unit cell are connected by impedance elements Z_l. All impedance elements are the same because the material is isotropic, and each element is associated with a unit vector $\underline{t}_l$ between the nodes. The second lattice is associated with admittance elements Y_s that are associated with the surface normals $\underline{n}_s$. The elements Y_s connect the mid-points of one unit cell to the mid-points of all contiguous ones.

The unit cell (and thus the sets of vectors $\underline{t}_l$ and $\underline{n}_s$) may be rotated arbitrarily with respect to an incident plane wave with wave direction $\hat{\underline{k}}$ and linear electric field polarisation $\hat{\underline{e}}$ orthogonal to $\hat{\underline{k}}$ (both defined as unit vectors).

Each of the elements Z_l represents a per unit length impedance in the limit as $D \rightarrow 0$, defined as the ratio of the curl of the electric field to the magnetic field in each of the three field directions, where the field components are evaluated on each of the impedance elements.

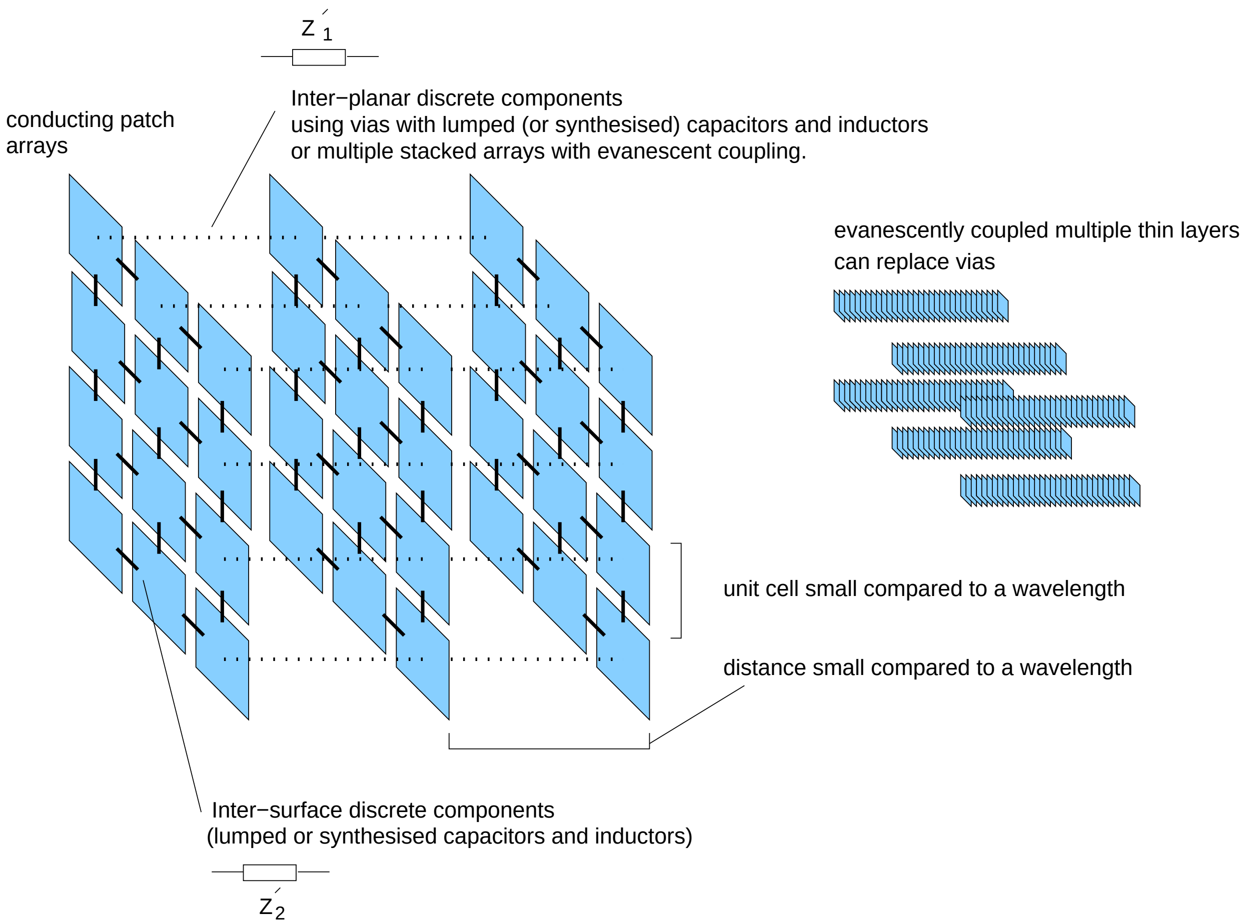

Figure 5: *A physical implementation of a laterally isotropic material.*

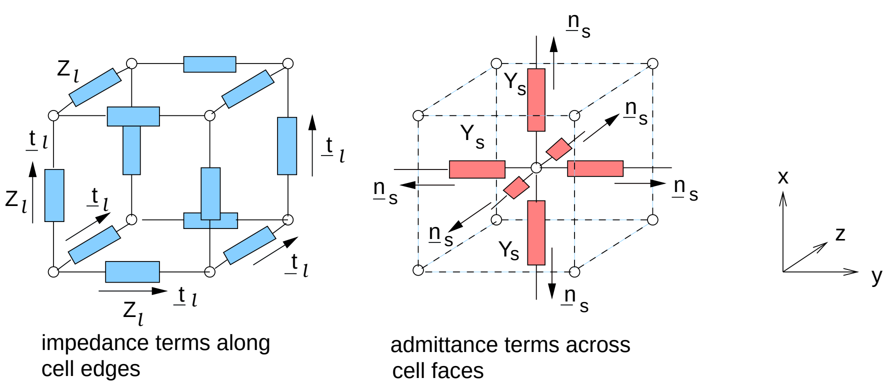

Figure 6: *Representation of an isotropic medium by a cubic unit cell of admittances and impedances.*

Similarly, each of the elements Y_s represent a per unit length admittance, defined as the ratio of the curl of the magnetic field to the electric field in each of the three field directions, with the field components evaluated on each of the admittance elements. The impedance edge elements are each shared between 4 unit cells, with a total of 12 edges so that there are three independent components per unit cell, one for each of the x, y and z-directions. Similarly, the surface elements are each shared between 2 unit cells, with a total of 6 surfaces so again there are three independent components per cell, one for each of the x, y and z-directions.

Because the three independent Z_l and Y_s are associated with orthogonal vector sets, $\underline{t}_l$ and $\underline{n}_s$, the incident plane wave field direction $\underline{e}$ always resolves into the same transmission line equation, independent of $\underline{e}$ or $\underline{k}$. This also implies that $Z(\omega) = Z_l$ and $Y(\omega) = Y_l$ in equations (14) and (13). This representation introduces no special restrictions on Z_l or Y_l, over and above what we have considered already.

Again, there are many possible physical implementations in just the same way that figure 5 was derived from figure 4. Figure 7 shows a candidate structure employing discrete (i.e. localised) components based on an array of perfectly conducting cubes. Similar examples have been proposed in the literature (see, e.g. [11]). This one controls Z_l, and hence Z_1', through the use of arrays of conducting patches between the cubes, where the discrete impedance elements are near the cube edges (as shown in figure 6). The elements Y_s, and hence Z_2', join contiguous conducting cubes. As far as we are aware this particular structure has not been trialed.

7 Homogeneity requirements

We now discuss a problem apparent as $D \to 0$. As remarked previously, both $Dp_0^{(1)}/c_0$ and $Dq_0^{(2)}/c_0$ must remain non-zero as $D \to 0$ in order that μ_r and the real part of ϵ_r remain bounded and independent of D. Here there is no difficulty since $Z_0/p_0^{(1)}$ represents the limiting value of the series inductive part of the lumped impedance $DZ(s)$ which is of order $O(D)$ as $D \to 0$. Similarly, $Z_0/q_0^{(2)}$ represents the limiting value of the shunt capacitive part of the lumped admittance $DY(s)$ which is also of order $O(D)$ as $D \to 0$.

However, if the network element $DZ(s)$ contains lumped capacitors, the values of the capacitors, $C_i^{(1)}$ are of order $O(1/D)$ and must become infinite if $Z(s)$ or $\mu_r(s)$ is constant as $D/\lambda_0 \to 0$. The same is true for $DY(s)$; i.e. if $DY(s)$ contains any inductors $L_i^{(2)}$ then these are of order $O(1/D)$ and must become infinite as $D/\lambda_0 \to 0$. In both cases, λ_0 refers to a wavelength, c_0/f, for a frequency f within a negative refractive index band.

When gyromagnetic materials are not present the ladder network values must be physically realised using real capacitors and real inductors; i.e. using charge storage and current storage mechanisms. This requires infinite value capacitors and inductors for all except conventional positive materials as the scale size tends to zero. Even when gyromagnetic materials are present the same remains true of the shunt admittance values modelling the permittivity dispersion. However, there are physical limitations set by achievable energy densities, non-linear

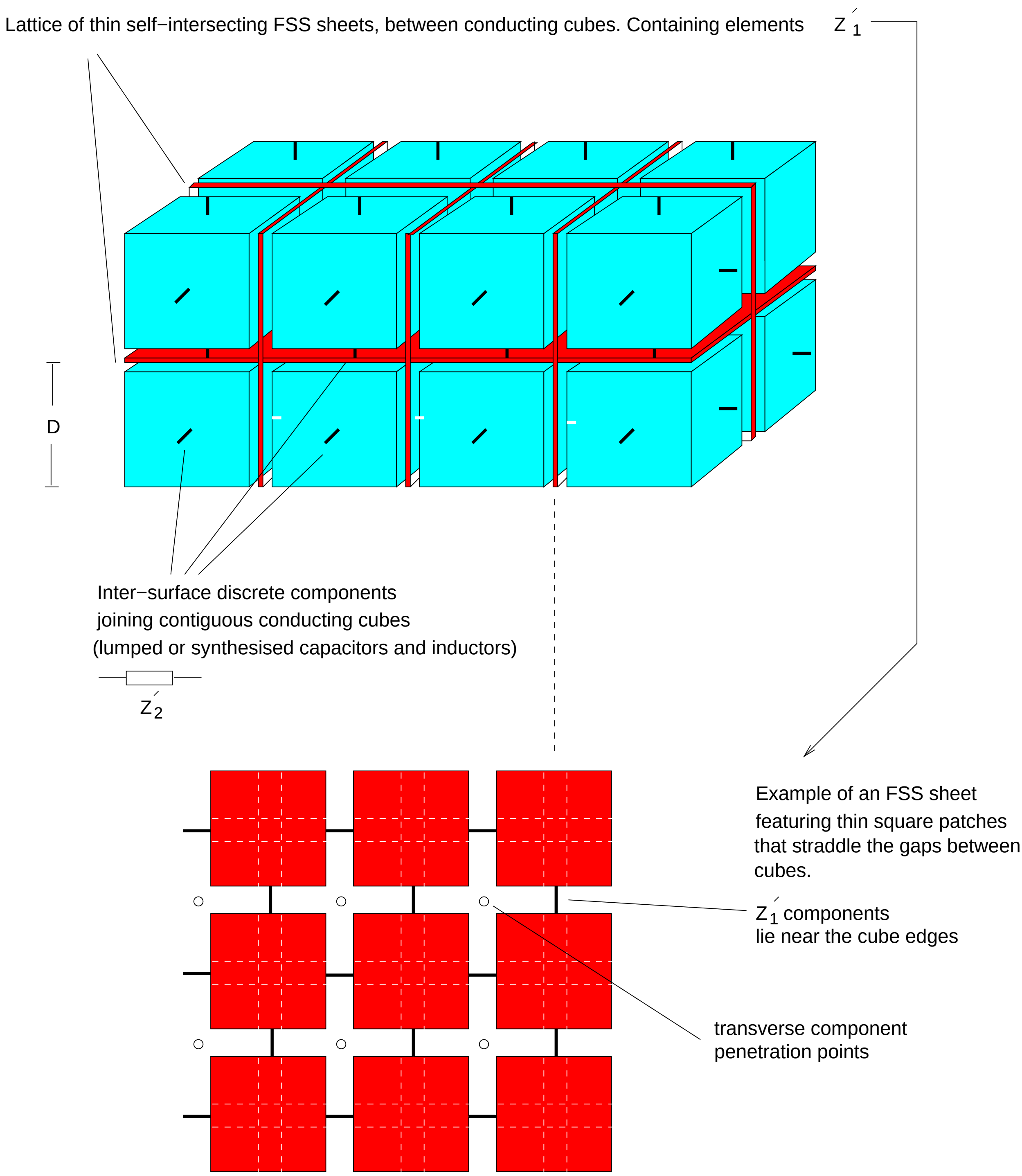

Figure 7: *A physical implementation of an isotropic material*

effects and parasitic capacitance and inductance on the maximum real inductance and capacitance that can be achieved per unit volume. These limits will set a non-zero lower bound to D in negative refractive index materials since for such materials both shunt inductors (inductors contained within $Y(s)$) and series capacitors (capacitors contained within $Z(s)$) are required.

The requirement for infinite capacitance and inductance, as $D \rightarrow 0$, does not *per-se* imply infinite energy density in the medium. For example with reference to figure 5, as $D \rightarrow 0$, the unit cell size tends to zero, so the current flowing through Z_2' tends to zero as the inductance tends to infinity. We believe, however, that this might be responsible for problems that occur in numerical modelling and numerical convergence in some moment-method schemes [16].

8 Some loss-bandwidth properties when losses are small

An advantage to the presented general forms for $\epsilon_r(\omega)$ and $\mu_r(\omega)$ is that they are amenable to simple analysis in the neighbourhood of the poles and zeros when losses are small. Let us first consider the effect of loss near the n_{th} pole in either $\epsilon_r(\omega)$ or $\mu_r(\omega)$ away from the origin. The effect of small loss is localised to the pole, so the pole term can be factored out from the remaining expression. For $\epsilon_r(\omega)$ we may thus write,

$$\epsilon_r(\omega) = F_n(\omega) g(\omega) \frac{c_0}{D} \frac{\prod_{i>0}(1-\omega/q_i^{(2)})(1+\omega/q_i^{(2)\star})}{\prod_{i>0,i\neq n}(1-\omega/p_i^{(2)})(1+\omega/p_i^{(2)\star})} \quad \text{for } |(\omega/p_n')^2 - 1| \ll 1 \tag{28}$$

where

$$F_n(\omega) = \frac{1}{(1-\omega/p_n^{(2)})(1+\omega/p_n^{(2)\star})} \tag{29}$$

The same factorisation may be applied to $\mu_r(\omega)$ with $p_n^{(2)} \rightarrow q_n^{(1)}$ and $g(\omega) \rightarrow 1/p_0^{(1)}$. $F_n(\omega)$ may now be expanded in the neighbourhood of $\omega = p_n'$. We find that the real part of $F_n(\omega)$, has a maximum and minimum when $\omega = \omega_m$ where,

$$\omega_m = p_n' \pm \Delta\omega \tag{30}$$

where

$$\Delta\omega = p_n'' + \frac{(p_n'')^3}{2(p_n')^2} + O((p_n'')^4) \tag{31}$$

for which

$$\Re(F_n(\omega_m)) = \frac{1}{4} \mp \left(\frac{p_n'}{4p_n''} + \frac{3p_n''}{8p_n'} \right) + O((p_n'')^2) \tag{32}$$

and

$$\Im(F_n(\omega_m)) = -\frac{p_n'}{4p_n''} + O((p_n'')^2) \tag{33}$$

To first order for small p_n'' we find that the loss tangent, $|\Re(\epsilon_r)/\Im(\epsilon_r)| \rightarrow 1$ when $\omega = \omega_m$. Similarly, if we define $\omega_b = p_n' + b\,\Delta\omega$ we find that

$$\left| \frac{\Re[\epsilon_r(\omega_b)]}{\Im[\epsilon_r(\omega_b)]} \right| \rightarrow 1/b \tag{34}$$

independent of p_n''. In other words, the product of the loss tangent and the frequency factor b is unity, independent of the loss at the pole, for small losses. The same is true for $\mu_r(\omega)$. If either ϵ_r or μ_r is bounded, such that $\Re(F_n(\omega_m))$ has maximum magnitude F_m, then for small p_n'' we require $p_n'' > p_n'/(4F_m)$.

9 A numerical example

Let us assume one of several relatively simple lossy dispersion characteristics suitable (we believe) for creation of a realisable negative refractive index material. The equivalent circuit for the characteristic impedance is as given in figure 4 (right hand side) where Z_1' is represented by a capacitor C_a, resistor R_a and inductor L_a in series, $Z_1' = R_a + sL_a + 1/sC_a$. The impedance element Z_2' is represented by an inductor L_b, a capacitor C_b and resistor R_b in series, all in parallel with another resistor R_2, $Z_2' = (R_b + sL_b + 1/sC_b) \,||\, R_2$. We will define all the circuit elements, at a scale D, by

$$
\begin{aligned}
C_a &= \frac{c_0}{(2\pi f_a)^2 Z_0 D} \quad R_a = r_a D \quad L_a = \frac{\eta_1 Z_0 D}{c_0} \quad L_1 = \frac{Z_0 D}{c_0} \\
L_b &= \frac{Z_0 c_0}{(2\pi f_b)^2 D} \quad R_b = \frac{1}{\sigma_b D} \quad C_b = \frac{\gamma \eta_2 D}{Z_0 c_0} \quad C_2 = \frac{\gamma D}{Z_0 c_0} \\
R_2 &= \frac{1}{\sigma_2 D}
\end{aligned}
\tag{35}
$$

where we assume $\mu_r(0) = 1$, $\Re(\epsilon_r(0)) = \gamma(1 + \eta_2) \geq 1$ and $\mu_r(\omega) > 0$ as $\omega \to \infty$.

The parameters f_a and f_b are characteristic frequencies of the medium in Hz, r_a is in Ωm^{-1}, σ_2 and σ_b are in $\Omega^{-1} m^{-1}$ and η_1 and η_2 are dimensionless. For simulation purposes, D is arbitrary.

Suppose $f_a = 3$ GHz, $f_b = 2.5$ GHz, $r_a = 100.0\,\Omega\, m^{-1}$, $\sigma_2 = 0.01\Omega^{-1} m^{-1}$, $\sigma_b = 1000.0\,\Omega^{-1}\, m^{-1}$, $\eta_1 = 0.5$ and $\gamma = \eta_2 = 1.0$. Figure 8 shows the real and imaginary parts of $\epsilon_r(f)$ and figure 9 the real and imaginary parts of $\mu_r(f)$, with f in GHz. The region where both real parts is negative lies between 2.500 GHz and 3.535 GHz. Over much of this region the loss is quite small. Figure 10 shows the electric and magnetic loss tangents which we define by $\Im(\epsilon_r)/|\Re(\epsilon_r)|$ and $\Im(\mu_r)/|\Re(\mu_r)|$, respectively. Finally, we plot the principal part of Stockman's integral $I_s = I_L + I_U$ computed as a function of δ between 0.01 and 0.0005. Simple quadrature is employed with uniform sampling using 10^5 and 5×10^5 samples per integral to ensure convergence to within a few percent over most of the range, except at the lower frequencies where convergence is worse. Convergence as $\delta \to 0$ is not achieved over the lower frequency range where I_s is positive. Whether convergence is achievable when $I_s < -1$ is not clear. This is shown in figures 11, 12 and 13 where the scale on the graphs is progressively changed. However, it would appear that $I_s(\delta) < -1$ for all computed values of frequency, f, between $f = 2.5$ to 3.5 GHz, consistent with our prediction of realisability. In addition we find that $I_s < -1$ between $f = 2.5$ to 3.9 GHz when $\delta = 0.0005$, though the material is not negative refractive between $f \approx 3.6$ to 3.9 GHz.

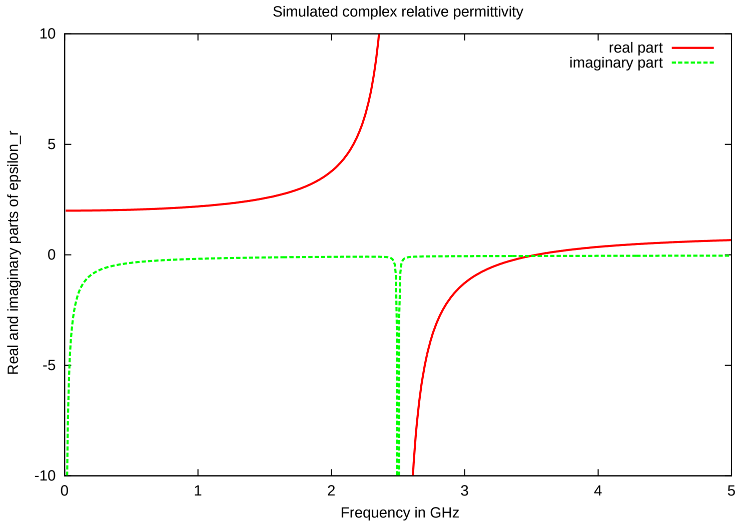

Figure 8: *Simulated relative permittivity, real and imaginary parts vs frequency*

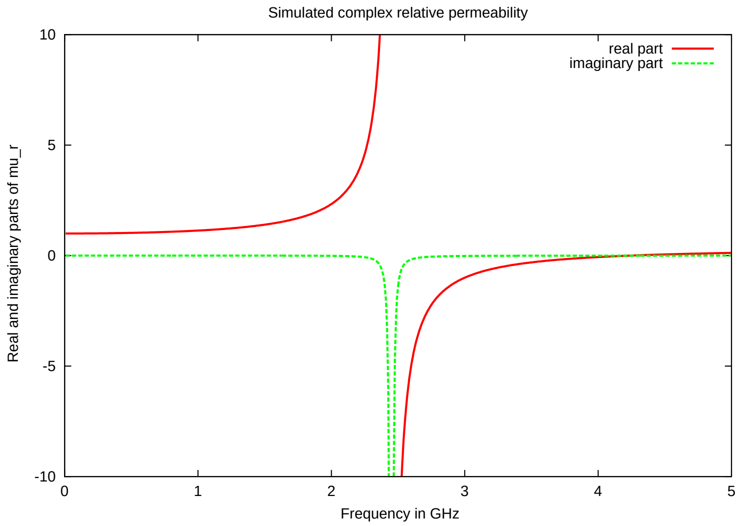

Figure 9: *Simulated relative permeability, real and imaginary parts vs frequency*

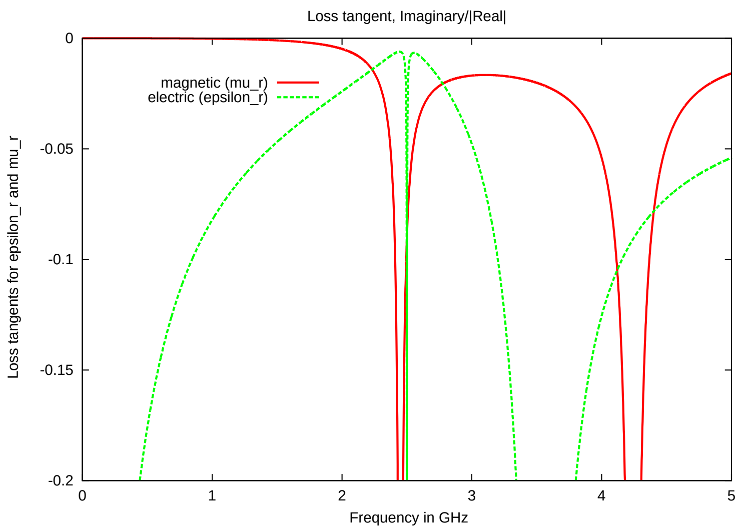

Figure 10: *Simulated electric and magnetic loss tangents vs frequency*

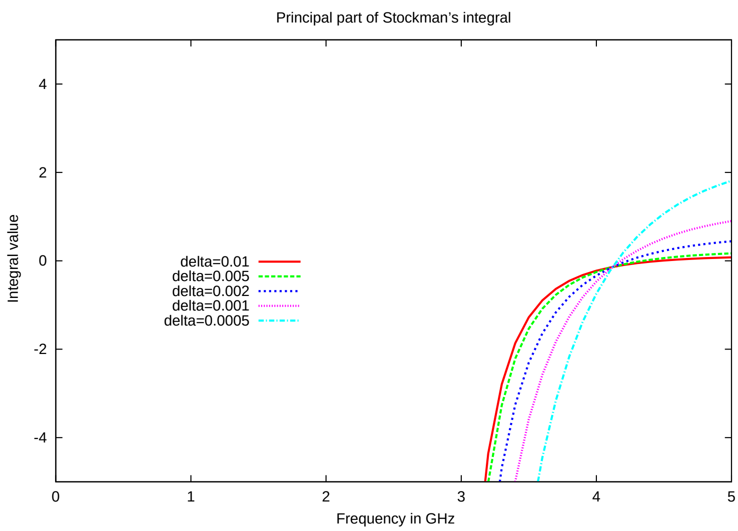

Figure 11: *Stockman integral (principal part) computed with various values of δ, scale 1.*

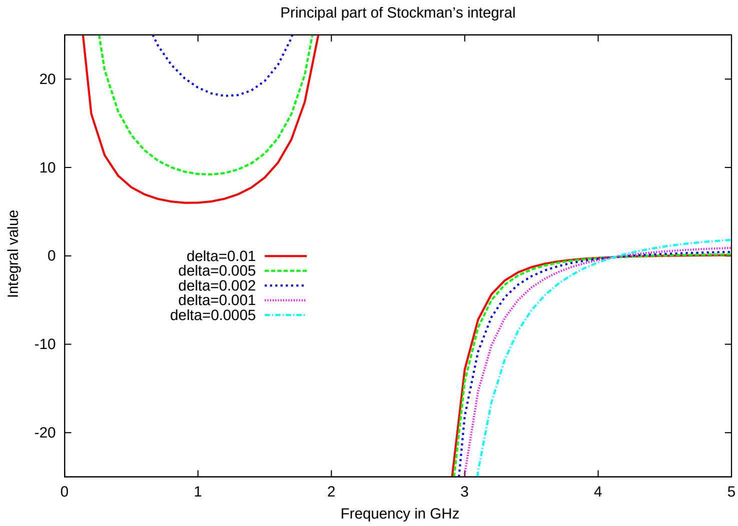

Figure 12: *Stockman integral (principal part) computed with various values of* δ*, scale 2.*

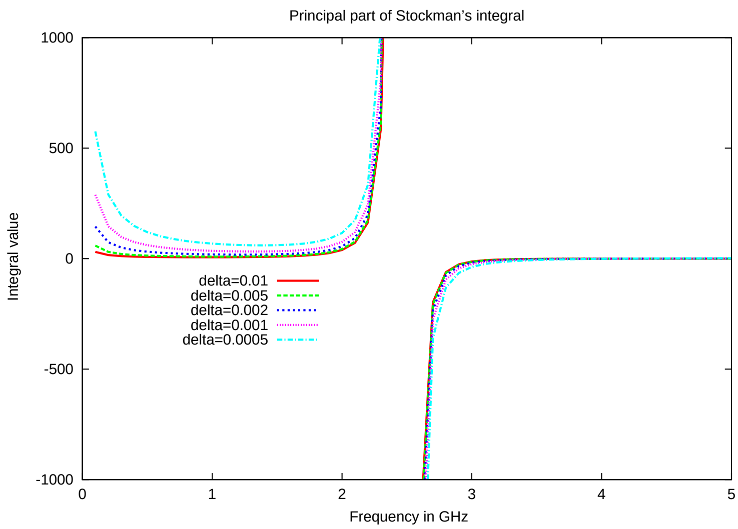

Figure 13: *Stockman integral (principal part) computed with various values of* δ*, scale 3.*

10 Conclusions

We have derived a general band structure for low loss quasi-homogeneous isotropic materials in terms of an equivalent circuit model with the correct low-frequency asymptotic properties. This follows as a direct result of Foster's theorem applied to a general positive rational function representation of $\epsilon_r(\omega)$ and $\mu_r(\omega)$. It follows from this model that negative refractive index materials may be constructed with arbitrary low loss only if there are frequencies (the pole frequencies) where $\epsilon_r(\omega)$ and $\mu_r(\omega)$ are arbitrarily large. We have shown our results are consistent with Stockman's integral bound [1], though there is a question regarding the existence of Stockman's integral as a finite measure principal value quantity.

We also point out that passive uniaxial or isotropic negative refractive index materials are impossible to fabricate in the homogeneous limit, defined as the scale length $D \to 0$, due to a requirement for indefinitely large capacitances and inductances when gyromagnetic materials are not present. This is true even if gyromagnetic materials are present since there is still a requirement for indefinitely large shunt inductors.

Consistent with Stockman's findings, we also show that any bound placed on the maximum magnitude of the relative permittivity and permeability introduces a loss in inverse proportion to the size of the bound, for small losses. The effect of the loss is quantified as a function of the difference in frequency from the band edge.

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